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Stability of the Triangular Equilibrium Points Influenced by Triaxial Primaries and Oblateness of Infinitesimal in the Elliptical Restricted Three Body Problem

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Abstract : This paper examines the effects of triaxiality of two primaries on the position and stability of the oblate infinitesimal motion about triangular equilibrium points $L_{(4,5)}$ in the framework of elliptical restricted three body problem. For determining the characteristic exponents of variational equations with periodic coefficients, we have used an analytical method, which is based on the Floquet's theory. The stability of infinitesimal around the triangular equilibrium points has been studied based on the analytical and numerical exploration which is simulated by drawing transition curves bounding the region of stability in the (μ-e) plane. The region of stability changed with variations in eccentricity, oblateness and triaxiality. It is observed that the equilibrium point is stable in the shaded portion of the transition curve, whereas it remains unstable outside the region of the transition curves.

Keywords: Elliptical Restricted Three Body Problem; Stability; Triaxiality; Oblate infinitesimal particle.

1. Introduction:

The present paper examines the effects of the Triaxiality of two primaries on the stability of the oblate infinitesimal body around the triangular equilibrium points of the elliptical restricted three-body problem (ER3BP). In the ER3BP models, the motion of an infinitesimal mass moves under the influence of two massive bodies revolving around their center of mass in an elliptical orbit but does not influence the two primaries. The eccentricity of the orbits plays a very important role, in the circular case it is missing Most of the celestial bodies orbit in elliptical rather than circularly Hence ER3BP describes the dynamical system more accurately. In this paper, we examine the stability of oblate infinitesimal around the triangular equilibrium points under the effects of the Triaxiality of primaries by exploiting an analytical method for determining the characteristic exponents, based on Floquet's theory. Danby(1964) studied the stability of the infinitesimal motion about the triangular points in the ER3BP; by using a digital computer numerical scheme based on Floquet theory result is given in the form of transition curves in μ–e plane. Bennet(1965) further studied the problem by taking five equilibrium points in elliptical restricted problem using characteristics exponents. The method adopted was same as employed by Moulton, F.R.(1914) in which he studied classical study of eight satellite of Jupiter. The above problem studied by Danby(1964) was further studied by Alfriend, K.T., Rand, R. H. (1969) by using two variable expansion method. The perturbation method based on Lie transform was further studied by Deprit, A., Rom, A. (1970) to develop power series of small parameter by characteristics exponents of the monodromy matrix They produced the principal parts of the characteristics exponents as general functions of mass ratio. The influence of eccentricity of the orbits of the primaries with or without radiation pressure on the existence of equilibrium points and their stability was studied by Zimvoschikov and Thakai (2004), Markeev (2005), and Ammar (2008). The influence of eccentricity, oblateness, and radiation parameters on the location and stability of collinear and triangular equilibrium points has been investigated by Narayan and Kumar (2011–2012). Recently, the linear stability of the triangular equilibrium points of the ER3BP has been studied by Narayan and Singh (2014a, 2014b) and Narayan and Usha (2014). The families of symmetric-periodic orbits in the three-dimensional elliptic problem with a variation of the mass ratio μ and the eccentricity e were studied by Sarris (1989). In the last year, Singh and Umar (2013) investigated the effects of the luminosity and oblateness of both primary bodies on the collinear libration points of the binary systems Achird, Luyten 726-8, Kruger 60, Alpha Centauri AB, and Xi Bootis moving in elliptic orbits around their common centre of mass. Recently, Singh and Umar (2014) have examined the collinear points of the ER3BP with a triaxial, bigger primary.

The present study is devoted to the analysis of the stability of oblate infinitesimal around triangular points under the triaxial primaries by exploiting the analytical technique developed by Bennet(1965). This method is based on Floquet’s theory for the determination of characteristics exponents for a system with periodic coefficients. The transition curves have been presented through the simulation technique, which shows the region of stability as well as instability for different values of triaxial parameters.

This paper is organized in five sections, section-1 describes introduction, section -2 provides the equations of motion, while section-3 describes the calculation of characteristic exponents and section-4 provides the graphical representation of transition curves, which are divided into stable and unstable regions. The discussion and conclusion are drawn in section-5.

2. Equation of motion

The differential equations of motion of the infinitesimal mass in the ER3BPunder triaxial primaries in a barycentric, pulsating system are given as: Poonam Duggad ,S. Dewangan ,A. Narayan(2020).

The differential equation of motion of the third body P in non dimensional barycentre, pulsating and non-uniformly rotating coordinate system (x, y) is written in the form:

$$x'' - 2y' = \frac{1}{1+e\cos v}\left(\frac{\partial \Omega}{\partial x}\right) \qquad y'' - 2x' = \frac{1}{1+e\cos v}\left(\frac{\partial \Omega}{\partial y}\right) \tag{2.1}$$

where ′ denotes differentiation with respect to v,

$$\Omega = \frac{x^2+y^2}{2} + \frac{1}{n^2}\left\{\frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{(1-\mu)[(2\sigma_1-\sigma_2)+A_4]}{2r_1^3} - \frac{3(1-\mu)[(\sigma_1-\sigma_2)y^2+A_4]}{2r_1^5} + \frac{\mu[(2\sigma_1'-\sigma_2')+A_4]}{2r_2^3} - \frac{3\mu[(\sigma_1'-\sigma_2')y^2+A_4]}{2r_2^5}\right\} \tag{2.2}$$

Where the mean motion of force due to oblateness given by Umar and Singh(2014)

$$n^2 = 1 + \frac{3}{2}e^2 + \frac{3}{2}(2\sigma_1 - \sigma_2) + (2\sigma_1' - \sigma_2') \tag{2.3}$$

and

$$r_1^2 = (x+\mu)^2 + y^2, r_2^2 = (x-1+\mu)^2 + y^2 \tag{2.4}$$

Where

$$r=\frac{a\left(1-e^2\right)}{\left(1+e\cos v\right)},\mu=\frac{m_2}{m_1+m_2},1-\mu=\frac{m_1}{m_1+m_2}$$

Where primaries masses are m_1 and m_2, where $\sigma_1,\sigma_2,\sigma_1^{'}, and \sigma_2^{'}$ are oblateness triaxial parameters ,while e is the eccentricity of orbits and v are is the true anomaly of the primaries respectively.

The coordinates of the triangular equilibrium points are

$$x=\frac{1}{2}-\mu-\frac{e^2}{2(1-\mu)}-\frac{\mu e^2}{2(1-\mu)}+\frac{e^2}{2}-\left(\frac{11}{8}+\frac{1-3\mu}{2\mu}\right)\sigma_1+\left(\frac{11}{8}+\frac{1-2\mu}{2\mu}\right)\sigma_2+\left(\frac{11}{8}-\frac{1}{1-\mu}+\frac{5\mu}{2(1-\mu)}\right)\sigma_1^{'}$$
$$+\frac{1}{2}\left(-\frac{7}{4}+\frac{1}{1-\mu}-\frac{3\mu}{(1-\mu)}\right)\sigma_2^{'}+\left(2-\frac{2}{(1-\mu)}+\frac{5\mu}{2(1-\mu)}\right)A_4$$

$$y=\pm\sqrt{\frac{3}{2}}\left[1-\frac{e^2}{3(1-\mu)}+\frac{e^2\mu}{3(1-\mu)}-\frac{e^2}{3}-\frac{2}{3}\left(\frac{11}{8}-\frac{1-3\mu}{2\mu}\right)\sigma_1+\frac{2}{3}\left(\frac{11}{8}-\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{2}{3}\left(-\frac{11}{8}-\frac{5\mu}{2(1-\mu)}+\frac{1}{1-\mu}\right)\sigma_1^{'}\right.$$
$$\left.+\frac{1}{3}\left(\frac{7}{4}-\frac{3\mu}{(1-\mu)}+\frac{1}{1-\mu}\right)\sigma_2^{'}-\frac{2}{3}\left(\frac{4}{2}-\frac{5\mu}{2(1-\mu)}+\frac{2}{1-\mu}\right)A_4\right] \tag{2.5}$$

Now the equation of motion around the equilibrium points L_4 are given by

$$\xi^{''}-2\eta^{'}=\Phi\left[\xi\Omega_{xx}^0+\eta\Omega_{xy}^0\right] \tag{2.6}$$
$$\eta^{''}-2\xi^{'}=\Phi\left[\xi\Omega_{yx}^0+\eta\Omega_{yy}^0\right] \tag{2.7}$$

Where ξ , η denote the small displacement in $\left(x_0,y_0\right)$

Then $x=x_0+\xi$ $\quad y=y_0+\eta$ (2.8)

Differentiating, we get:

$x^{'}=\xi^{'},y^{'}=\eta^{'}$ and $x^{''}=\xi^{''},y^{''}=\eta^{''}$

(2.9)

Where, $\Phi=\dfrac{1}{1+e\cos v}$ (2.10)

The coordinates of equilibrium points are given by

$$x_0=\frac{1}{2}-\mu-\frac{e^2}{2(1-\mu)}-\frac{\mu e^2}{2(1-\mu)}+\frac{e^2}{2}-\left(\frac{11}{8}+\frac{1-3\mu}{2\mu}\right)\sigma_1+\left(\frac{11}{8}+\frac{1-2\mu}{2\mu}\right)\sigma_2+\left(\frac{11}{8}-\frac{1}{1-\mu}+\frac{5\mu}{2(1-\mu)}\right)\sigma_1^{'}+$$
$$\frac{1}{2}\left(-\frac{7}{4}+\frac{1}{1-\mu}-\frac{3\mu}{(1-\mu)}\right)\sigma_2^{'}+\left(2-\frac{2}{(1-\mu)}+\frac{5\mu}{2(1-\mu)}\right)A_4$$

$$y_0=\pm\sqrt{\frac{3}{2}}\left[\frac{1}{1}-\frac{e^2}{3(1-\mu)}+\frac{e^2\mu}{3(1-\mu)}-\frac{e^2}{3}-\frac{2}{3}\left(\frac{11}{8}-\frac{1-3\mu}{2\mu}\right)\sigma_1+\frac{2}{3}\left(\frac{11}{8}-\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{2}{3}\left(-\frac{11}{8}-\frac{5\mu}{2(1-\mu)}+\frac{1}{1-\mu}\right)\sigma\right.$$
$$\left.+\frac{1}{3}\left(\frac{7}{4}-\frac{3\mu}{(1-\mu)}+\frac{1}{1-\mu}\right)\sigma_2^{'}-\frac{2}{3}\left(2-\frac{5\mu}{2(1-\mu)}+\frac{2}{1-\mu}\right)A_4\right] \tag{2.11}$$

Now, differentiating Ω partially with respect to x, y respectively and evaluating Ω_{xx}, Ω_{xy} and Ω_{yy} at the equilibrium points (x_0,y_0) [Refer Appendix-1]

Now transforming equation (2.6) and (2.7) in matrix form, we get

$$X^{'}=PX \tag{2.12}$$

Where,

$$X = \begin{bmatrix} \xi \\ \eta \\ \xi' \\ \eta' \end{bmatrix} \quad , \quad X' = \begin{bmatrix} \xi' \\ \eta' \\ \xi'' \\ \eta'' \end{bmatrix} \tag{2.13}$$

$$P(v,e) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \phi\Omega_{xx} & \phi\Omega_{xy} & 0 & 2 \\ \phi\Omega_{yx} & \phi\Omega_{yy} & -2 & 0 \end{bmatrix} \tag{2.14}$$

3. Determination of characteristics exponents

Exploiting the Floquet's theory and using the theoretical method developed by Bennett et al.(1965), we shall determine the characteristic exponents. For applying the Floquets theory the solution of the system of equation (2.12) is written in the form:

$$X_k = Y_k e^{\lambda_k v} \tag{3.1}$$

Where Y_k is periodic coefficient with period 2π and λ_k is the characteristics exponents (2.15).

$$X = Y e^{\lambda v} \tag{3.2}$$

Now, Dropping the suffix for the solution in general form, the derivative of the solution with respect to v is is given as:

$$Y' = (P - \lambda I) Y \tag{3.3}$$

Where I is the unit matrix. Now taking the expansion of the coefficient Y, the characteristic exponent λ and the matrix P in terms of the eccentricity of orbit e as: $Y = Y^{(0)} + eY^{(1)} + e^2 Y^{(2)} + \ldots\ldots$

$$\lambda = \lambda_0 + e\lambda_1 + e^2\lambda_2 + \ldots\ldots \tag{3.4}$$

And matrix P is expanded as: $P(v,e) = P^{(0)} + eP^{(1)} + e^2 P^{(2)} + \ldots\ldots$ (3.5)

Where, $$P^{(0)} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \Omega^0_{xx} & \Omega^0_{xy} & 0 & 2 \\ \Omega^0_{yx} & \Omega^0_{yy} & -2 & 0 \end{bmatrix} \tag{3.6}$$

and $$P^{(m)} = (-\cos v)^m C \; ; \; m = 1, 2, 3\ldots. \tag{3.7}$$

Such that $$C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Omega^0_{xx} & \Omega^0_{xy} & 0 & 0 \\ \Omega^0_{yx} & \Omega^0_{yy} & 0 & 0 \end{bmatrix} \tag{3.8}$$

Substituting the value of Y, Y', and P from equation (3.4) and equation (3.5) in equation (3.3), we get:

$$Y'^{(0)} + eY'^{(1)} + e^2 Y'^{(2)} + \ldots\ldots = \begin{bmatrix} \left\{ (P'^{(0)} + eP'^{(1)} + e^2 P'^{(2)} + \ldots\ldots) - I(\lambda_0 + e\lambda_1 + e^2\lambda_2 + \ldots.) \right\} \\ .(Y'^{(0)} + eY'^{(1)} + e^2 Y'^{(2)} + ..) \end{bmatrix} \tag{3.9}$$

Equating the coefficient of terms with the same power in e from both sides and using equation (3.4), we obtain the following equations

$$Y'^{(0)} + [I\lambda_0 - P^{(0)}] Y^{(0)} = 0 \qquad Y'^{(1)} + [I\lambda_0 - P^{(0)}] Y^{(1)} = (-C\cos v - I\lambda_1) Y^{(0)}$$

$$Y^{'(2)} + \left[I\lambda_0 - P^{(0)}\right]Y^{(2)} = \left(-C\cos v - I\lambda_1\right)Y^{(1)} + \left(C\cos^2 v - I\lambda_2\right)Y^{(0)}$$

Proceeding further , we get

$$Y^{'(n)} + \left[I\lambda_0 - P^{(0)}\right]Y^{(n)} = \sum_{m=1}^{n}\left(-C(\cos v)^m - I\lambda_m\right)Y^{(n-m)} \tag{3.10}$$

The solution of the zeroth order equation will be a constant vector and the n^{th} order equation with non-homogeneous terms have frequencies upto $\frac{n}{2\pi}$. Then the particular solution is given as

$$Y^{(n)} = \sum_{k=-n}^{k=n} a^{(n,k)} e^{ikv}; \quad n = 0;1;2; \tag{3.11}$$

Where, $$a^{(n,k)} = \begin{bmatrix} a_1^{(n,k)} \\ a_2^{(n,k)} \\ a_3^{(n,k)} \\ a_4^{(n,k)} \end{bmatrix} \tag{3.12}$$

Substituting the values from equation (3.11) and (3.12) in equation (3.10), we obtain from the first equation of system equation (3.10)

$$\left[I\lambda_0 - P^0\right]a^{(0,0)} = 0 \tag{3.13}$$

$$\left[I\lambda_0 - P^0\right]a^{(1,0)} = -\lambda_1 a^{(0,0)} \tag{3.14}$$

$$\left[I(\lambda_0 + i) - P^0\right]a^{(1,1)} = -\frac{1}{2}Ca^{(0,0)} \tag{3.15}$$

$$\left[I(\lambda_0 - i) - P^0\right]a^{(1,-1)} = -\frac{1}{2}Ca^{(0,0)} \tag{3.16}$$

$$\left[I\lambda_0 - P^0\right]a^{(2,0)} = -\lambda_1 a^{(1,0)} + \left[-\frac{C}{2} - I\lambda_2\right]a^{(0,0)} - \frac{C}{2}\left(a^{1,-1} + a^{(1,1)}\right) \tag{3.17}$$

For the existence of $a^{(0,0)}$, it is necessary that

$$\det\left(I\lambda_0 - P^0\right) = 0 \tag{3.18}$$

That is, $$\begin{bmatrix} \lambda_0 & 0 & -1 & 0 \\ 0 & \lambda_0 & 0 & -1 \\ -\Omega_{xx}^0 & \Omega_{xy}^0 & \lambda_0 & -2 \\ -\Omega_{yx}^0 & -\Omega_{yy}^0 & 2 & 0 \end{bmatrix} = 0$$

From the above relation, we get:

$$\lambda_0^4 + \left(4 - \Omega_{xx}^0 - \Omega_{yy}^0\right)\lambda_0^2 + \left\{\Omega_{xx}^0\Omega_{yy}^0 - \left(\Omega_{xy}^0\right)^2\right\} = 0$$

$$\therefore\ \lambda_0^4 - Q\lambda_0^2 + R = 0 \tag{3.19}$$

where, $$Q = \Omega_{xx}^0 + \Omega_{yy}^0 - 4 \tag{3.20}$$

$$R = \Omega_{xx}^0\Omega_{yy}^0 - \left(\Omega_{xy}^0\right)^2 \tag{3.21}$$

Solving equation (3.19), we obtain the value of λ_0 as follows:

$$\lambda_0^2 = \frac{Q \pm \left(Q^2 - 4R\right)^{\frac{1}{2}}}{2} \tag{3.22}$$

Substituting the values of $\Omega^0_{xx}, \Omega^0_{yy} and \Omega^0_{xy}$ in the equation (3.20) and equation (3.21), for the value of Q and R ,refer Appendix-II

From the first equation of the system of equations (3.13), it can be observed that it is necessary that the determinant of the coefficients on the left with any column replaced by the non-homogenous terms on the right be zero.

$$\left[I\lambda_0 - P^{(0)}\right]a^{(1,0)} = -\lambda_1 a^{(0,0)} \tag{3.23}$$

That is $\det\left(I\lambda_0 - P^{(0)}\right)a^{(1,0)} + \lambda_1 a^{(0,0)}$ vanishes. Since λ_1 appears as a factor in each term, therefore we get

$$\lambda_1 \det\left[I\lambda_0 - P^{(0)}\right] + a^{(0,0)} = 0 \tag{3.24}$$

But $\det\left[I\lambda_0 - P^{(0)}\right] \neq 0$ (3.25)

Therefore $\lambda_1 = 0$

Again from Equations (3.15) and (3.16), the solutions for $a^{(1,1)}$ and $a^{(1,-1)}$ are :

$$a^{(1,1)} = -\frac{1}{2}\left[I(\lambda_0 + i) - P^{(0)}\right]^{-1} Ca^{(0,0)} \qquad a^{(1,-1)} = -\frac{1}{2}\left[I(\lambda_0 - i) - P^{(0)}\right]^{-1} Ca^{(0,0)} \tag{3.26}$$

Substituting the value of $a^{(1,1)}$ and $a^{(1,-1)}$ equation (3.17) can also be written as:

$$\left[I\lambda_0 - P^{(0)}\right]a^{(2,0)} = \frac{1}{4}\left[\left(I(\lambda_0 + i) - P^{(0)}\right)^{-1} + \left(I(\lambda_0 - i) - P^{(0)}\right)^{-1}\right]Ca^{(0,0)} + \left(\frac{C}{2} - I\lambda_2\right)a^{(0,0)} \tag{3.27}$$

The matrices with in the square bracket are complex conjugates, so that only real parts of either needs to be considered and then equation (3.27) can be written as:

$$\left[I\lambda_0 - P_0\right]a^{(2,0)} = \left[\frac{1}{2}C\,\mathrm{Re}\left\{\left(I(\lambda_0 + i) - P^{(0)}\right)\right\}^{-1}\right] + \left(\frac{1}{2}C - I\lambda_2\right)a^{(0,0)} \tag{3.28}$$

Now, after some mathematical manipulations the value of λ_2 can be obtained from equations (3.28) given by:

$$\lambda_2 = -\frac{\left(Q^2 - 4R - 16\right)\lambda_0^2 + A_0F_0 + A_1F_1 + A_2F_2}{4\left(Q^2 - 4Q - 4R\right)\lambda_0^2 + 32R} \tag{3.29}$$

Or $\lambda_2 = A\lambda_0$ (3.30)

Where
$$A = -\frac{\left(Q^2 - 4R - 16\right)\lambda_0^2 + A_0F_0 + A_1F_1 + A_2F_2}{4\left(Q^2 - 4Q - 4R\right)\lambda_0^2 + 32R}$$

$$A_0 = \left[(Q+4)^2(Q-4) - 4QR\right]\lambda_0^2 - R(Q+4) - 4R^2$$

$$F_0 = \frac{1}{N}\left[(Q+1)\lambda_0^2 + (Q+1) + 2R\right] \qquad A_1 = -8\lambda_0 R\left[2\lambda_0^2 - (Q+4)\right]$$

$$F_1 = -\frac{\lambda_0}{N}\left[2\lambda_0^2 + (Q+3)\right] \qquad A_2 = -8\lambda_0^2 R\left(Q^2 - 4R - 16\right)$$

$$F_2 = -\frac{1}{N}\left[\lambda_0^2 - (Q+1)\right] \qquad N = \lambda_0^2(-16R) - 4R \tag{3.31}$$

Now substituting all the above value in equation (3.30), the value of A is obtained as:

$$A = -\frac{T}{D} \tag{3.32}$$

where, $$T = \left(Q^2 - 4R - 16\right)\lambda_0^2 + A_0F_0 + A_1F_1 + A_2F_2 \tag{3.33}$$

and $$D = 4\left(Q^2 - 4Q - 4R\right)\lambda_0^2 + 32R \tag{3.34}$$

Thus, the characteristics exponents up to second order of approximation in e can be written as:

$$\lambda = \lambda_0 + e^2\lambda_2$$

4. Transition curves separating stable and unstable regions.

The transition curves describe the stability of the triangular equilibrium points in the ERTBP by separating stable and unstable regions. The transition curves separating the stable and unstable regions can be found by simply equating the expression for the characteristics roots or exponents to the value for periodic solution in the range $0 \le \mu \le \frac{1}{2}$.

From Floquet's theory, if λ_k are characteristics exponents then in polar form it can be written as:

$$\lambda_k = \frac{1}{T}\left[L_N b_k + i(\theta_k + 2n\pi)\right], n = 0, \pm 1, \pm 2 \tag{4.1}$$

That is, the characteristics exponents are only determined within the imaginary multiple of $\frac{2n\pi}{T}$, but it is the real part of the exponent that determines if the solution is bounded or not. Hence, the corresponding periodic solution form of equation (4.1) is obtained as:

$$\lambda = i\left(n \pm \frac{1}{2}\right), n = 0, \pm 1, \pm 2 \tag{4.2}$$

In the range $0 \le \mu \le \frac{1}{2}$, the periodic solution is given by: $\lambda^* = \pm\frac{i}{2}$

Replacing λ by λ^* in equation (3.35), we have: $\pm\frac{i}{2} = \lambda_0 + e^2\lambda_2 = \left(1 + e^2 A\right)\lambda_0$ Simplifying the above equation further we get:

$$e^2 = \frac{1}{A}\left[\pm\left(-\frac{1}{4\lambda_0^2}\right)^{\frac{1}{2}}\right] \tag{4.3}$$

Now, evaluating the values of A from equation (3.32) and that of λ_0^2 from equation (3.22), the values of e can be calculated for different value of μ .

Using equation (4.3) we get the relation between μ and e , which gives the transition curves in the $\mu - e$ plane.

5. Discussion and Conclusion

Based on the Floquet's theory, the stability of oblate infinitesimal's motions about the triangular equilibrium points in the ER3BP with triaxial primaries has been investigated using analytical technique. The transition curves in the μ - e plane has been plotted in Fig 1-3 ,it was found that when all the other perturbing forces were present then the value of μ lies between .05 to .06 was found same. From Fig 4-9 ,it was found that when all the other perturbing forces are neglected then also the value of μ lies between .05 to .06.

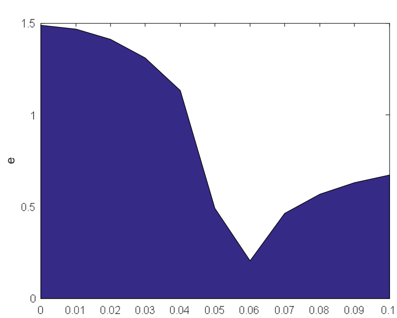

Figure 1. The transition Curves in μ -e plane for ε_1=0.0009; ε_2 =0.0004; σ_1=0.0002; σ_2=0.0009; σ_1'=0.0002; σ_2'=0.0008; A_4=0.0002

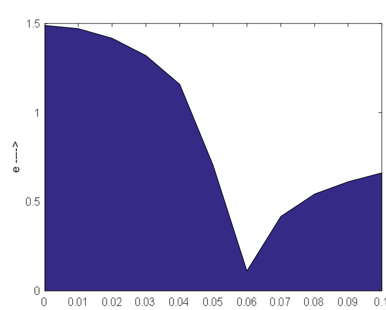

Figure 2. The transition Curves in μ - e plane for ε_1=0.0004; ε_2 =0.0002; σ_1=0.0009; σ_2=0.0005; σ_1'=0.0007; σ_2'=0.0003; A_4=0.0005

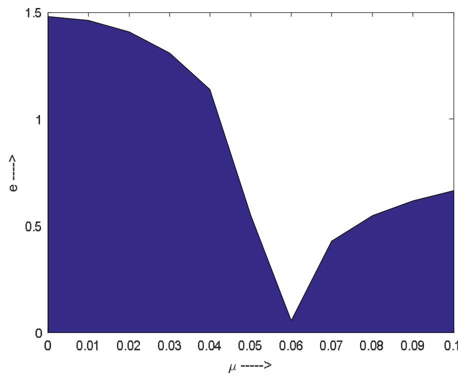

Figure 3. The transition Curves in μ - e plane for ε_1=0.0001; ε_2=0.0002; σ_1=0.0007; σ_2=0.0001; σ_1'=0.0005; σ_2'=0.0002; A_4=0.0009

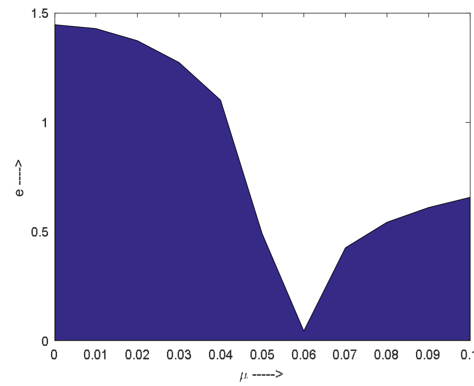

Figure 4. The transition Curves in μ - e plane for ε_1=0; ε_2=0; σ_1=0; σ_2=0; σ_1'=0; σ_2'=0; A_4=0.0005

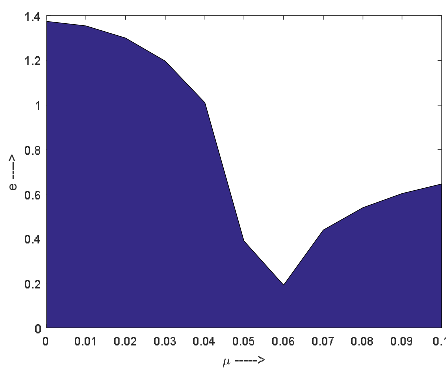

Figure 5. The transition Curves in μ - e plane for ε_1=0.0008; ε_2=0; σ_1=0; σ_2=0; σ_1'=0; σ_2'=0.; A_4=0

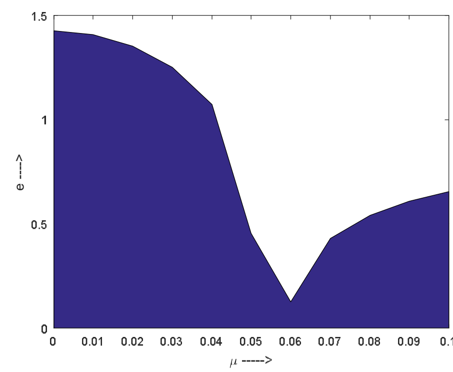

Figure 6. The transition Curves in μ - e plane for ε_1=0; ε_2=0.0001; σ_1=0; σ_2=0; σ_1'=0; σ_2'=0.; A_4=0

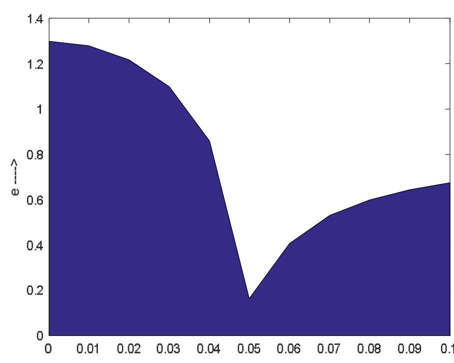

Figure 7. The transition Curves in μ - e plane for ε_1=0; ε_2=0; σ_1=0.0005; σ_2=0.0001; σ_1'=0; σ_2'=0; A_4=0

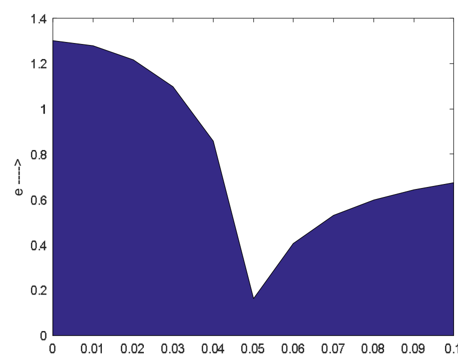

Figure 8. The transition Curves in μ - e plane for ε_1=0; ε_2=0; σ_1=0; σ_2=0; σ_1'=0.0005; σ_2'=0.0001; A_4=0

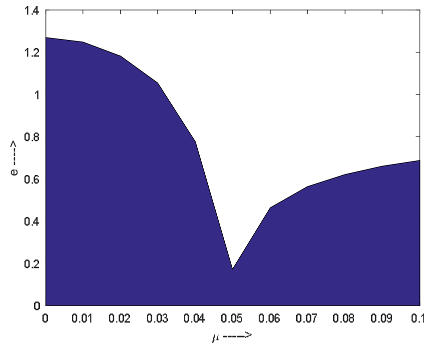

Figure 9. The transition Curves in μ - e plane for

$$\varepsilon_1=0; \varepsilon_2=0; \sigma_1=0.0004; \sigma_2=0.0002; \sigma_1^{'}=0.0007; \sigma_2^{'}=0.0002; A_4=0$$

References

[1] Ammar, M.K., The effect of solar radiation pressure on the Lagran-gian points in the elliptical restricted three body problem, Astro-physics and Space Science, 313 , 393- 408(2008) https://doi.org/10.1007/s10509-007-9709-z

[2] Alfriend, K.T., Rand, R. H.: Stability of the triangular points in the elliptic restricted problem of three bodies. AIAAJ 6, 1024(1969) https://doi.org/10.2514/3.5270

[3] Aishetu Umar, JagadishSingh(2014) "Periods, Eccentricities and Axes around $L_{4,5}$ in the ER3BP under Radiating and Oblate Primaries" International Journal of Astronomy and Astrophysics Vol.04 No.04 (2014), Article ID:52771, 14 pages https://doi.org/10.4236/ijaa.2014.44061

[4] Bennett, A.: In Astrodynamicssecialist Conference. Monetary, Cali-fornia. 101 (1965)

[5] Bennett, A.: Characteristic exponents of the five equilibrium solu-tions in the Elliptically Restricted Problem, ICARUS 4, 177-187(1965) https://doi.org/10.1016/0019-1035(65)90060-6

[6] Danby JMA.: Stability of the triangular points in the elliptical re-stricted problem of three bodies, Astronomical Journal, 69,165-172 (1964) https://doi.org/10.1086/109254

[7] Deprit, A., Rom, A.: Characteristics exponents at L4 in the elliptic restricted problem. Astron. Astrphysc.5, 416-425(1970)

[8] Floquet, G., 1883. Sur les equations differentielleslinearires a coefficients periodiques.Annales de I'EcoleNormaleSuperieure 12,47-88. http://doi.org/10.24033/asens.220

[9] Markeev A.P. (2005)" One special case of parametric resonance in problem of celestial mechanics" Astronomy letter vol31 , No. 5,pp 300-356, 2005. http://doi.org/10.1134/1.1922534

[10] Moulton, F.R.: On the stability of direct and retrograde orbits. International Journal of Advanced Astronomy As-tron.Soc. 75, 40-57 (1914) https://doi.org/10.1093/mnras/75.2.40

[11] Narayan, A., Kumar, C.R. Effects of photogravitational and oblate-ness on the Triangular Lagrangian points in the Elliptical Restricted three body problem, Indian Journal of Pure and Applied Mathemat-ics,68, No.2, 201-224(2011)

[12] Narayan, A., Singh, N.: Motion and stability of triangular equilibri-um points in elliptical restricted three body problem under the radi-ating primaries. Astrophys.Space Sci.352 (1), 57-70 (2014) https://doi.org/10.1007/s10509-014-1903-1

[13] Narayan, A., Usha, T., Stability of triangular equilibrium points in the elliptic restricted problem of three bodies with radiation and tri-axial primaries, Astrophysics and Space Science,351(1), 135-142(2014) https://doi.org/10.1007/s10509-014-1818-x

[14] Narayan,A., Singh, N.: Stability of triangular points in elliptical restricted three body problem under the radiating binary system . Astrophys. Space Sci. http://doi.org/10.1007/s 10509-014-2014-8

[15] Poonam D.,Dewangan.S ,,Narayan.A :Effects of triaxiality of primaries on oblate infinitesimal in elliptical restricted three body problem. New Astronomy http://doi.org/10.1016/j.newast.2020.101538

[16] Singh, J. and Umar, A. (2013) Collinear Equilibrium Points in the Elliptic R3BP with Oblateness and Radiation.Advances in Space Research,52, 1489-1496. http://doi.org/10.1016/j.asr.2013.07.027

[17] Singh, J. and Umar, A. (2014) On Motion around the Collinear Libration Points in the Elliptic Restricted Three-Body Problem with a Bigger Triaxial Primary. New Astronomy,29,36-41. http://doi.org/10.1016/j.newast.2013.11.003

[18] Zimvoschikov, A.S., Thkai, V.N., Instability of libration points and resonance phenomena in the photogravitational elliptical restricted three body problem, Solar system research, 38, No.2, 155-163(2004). https://doi.org/10.1023/B:SOLS.0000022826.31475.a7

Supplementary materials

Appendix-I

$$
\begin{aligned}
\Omega^0_{xx} = {} & \frac{3}{8}+\frac{9\varepsilon_1}{8}-\frac{9\mu\varepsilon_1}{8}+\frac{9\mu\varepsilon_2}{8}+\frac{27e^2}{16}-\frac{3\mu e^2}{2}-\frac{3e^2}{4(1-\mu)}+\frac{15e^2\varepsilon_1}{4(1-\mu)}-\frac{27e^2\varepsilon_1}{16}+\frac{3\mu e^2}{4(1-\mu)}+ \\
& \Big[\frac{207}{64}+\frac{5\mu}{16}-\frac{69\varepsilon_1}{16}+\frac{489\mu\varepsilon_1}{16}-\frac{309\mu\varepsilon_2}{16}-\frac{219e^2}{32}+\frac{63e^2}{8(1-\mu)}-\frac{3}{4}\left(\frac{1-3\mu}{\mu}\right)+\frac{15}{4}\left(\frac{1-3\mu}{\mu}\right)\varepsilon_1+\frac{3}{4}(1-3\mu) \\
& -\frac{15}{4}(1-3\mu)\varepsilon_1\Big]\sigma_1+\Big[\frac{63}{64}+\frac{15\mu}{16}+\frac{477\varepsilon_1}{16}+\frac{1587\mu\varepsilon_1}{64}+\frac{237\mu\varepsilon_2}{16}+\frac{159e^2}{16}+\frac{15}{4}(1-2\mu)\varepsilon_2+\frac{3e^2}{32(1-\mu)}+ \\
& \frac{3}{4}\left(\frac{1-2\mu}{\mu}\right)+\frac{15}{4}\left(\frac{1-2\mu}{\mu}\right)\varepsilon_1-\frac{3}{2}(1-2\mu)+\frac{15}{4}(1-2\mu)\varepsilon_1\Big]\sigma_2+\Big[\frac{33}{4}-\frac{411\mu}{64}-\frac{21\varepsilon_1}{2}+\frac{219\mu\varepsilon_1}{16}-\frac{18801\mu\varepsilon_2}{64}-\frac{105\varepsilon_2}{4}- \\
& \frac{591e^2}{32}+\frac{12e^2}{(1-\mu)}-\frac{3}{2}\left(\frac{1}{1-\mu}\right)+\frac{27}{4}\left(\frac{\mu}{1-\mu}\right)+\frac{15}{2}\left(\frac{\varepsilon_1}{1-\mu}\right)-\frac{37}{4}\left(\frac{\mu\varepsilon_1}{1-\mu}\right)-\frac{15\mu^2}{2(1-\mu)}+\frac{3\mu(1-3\mu)}{4}-\frac{15}{2}\left(\frac{\mu\varepsilon_2}{1-\mu}\right)\Big]\sigma_1'+ \\
& \Big[-6+\frac{275\mu}{64}+\frac{33\varepsilon_1}{4}+\frac{3\mu\varepsilon_1}{4}+\frac{105\varepsilon_2}{4}+\frac{19233\mu\varepsilon_2}{64}+\frac{375e^2}{8}-\frac{87e^2}{8(1-\mu)}+\frac{3}{4(1-\mu)}-\frac{15}{4}\left(\frac{\mu}{1-\mu}\right)-\frac{15}{4}\left(\frac{\varepsilon_1}{1-\mu}\right)+ \\
& \frac{15\mu\varepsilon_1}{1-\mu}+\frac{9\mu^2}{2(1-\mu)}+\frac{15}{4}\left(\frac{\mu\varepsilon_2}{1-\mu}\right)\Big]\sigma_2'+\Big[\frac{21}{4}-\frac{1065\varepsilon_1}{16}-\frac{9\mu}{2}-\frac{105\varepsilon_2}{8}+\frac{345\mu\varepsilon_2}{16}-\frac{1425e^2}{64}+\frac{69e^2}{4(1-\mu)}-3\left(\frac{1}{1-\mu}\right)+ \\
& \frac{39}{4}\left(\frac{\mu}{1-\mu}\right)+15\left(\frac{\varepsilon_1}{1-\mu}\right)-\frac{135}{4}\left(\frac{\mu\varepsilon_1}{1-\mu}\right)-\frac{15\mu^2}{2(1-\mu)}+\frac{15\mu\varepsilon_1}{2}+15\mu\varepsilon_2-\frac{5\mu\varepsilon_2}{1-\mu}\Big]A_4
\end{aligned}
$$

$$
\begin{aligned}
\Omega^0_{xy} = {} & \frac{3\sqrt{3}}{2}\Big\{\frac{1}{2}-\mu-\frac{5\varepsilon_1}{2}+\frac{5\mu\varepsilon_1}{2}+\frac{5\mu\varepsilon_2}{2}-\frac{5e^2}{12}-\frac{2e^2}{3(1-\mu)}+\frac{10e^2\varepsilon_1}{3(1-\mu)}+\frac{25e^2\varepsilon_1}{12}+\frac{11\mu e^2}{6}+ \\
& \Big[\frac{31}{24}-\frac{21\mu}{4}-\frac{359\varepsilon_1}{24}-\frac{481\mu\varepsilon_1}{24}-\frac{35\mu\varepsilon_2}{12}-\frac{119e^2}{48}-\frac{13e^2}{4(1-\mu)}+\frac{1}{3}\left(\frac{1-3\mu}{2\mu}\right)+\frac{4e^2}{3}\left(\frac{1-3\mu}{2\mu}\right)+\frac{10\varepsilon_1}{3}\left(\frac{1-3\mu}{2\mu}\right)+ \\
& \frac{\mu}{3}\left(\frac{1-3\mu}{2\mu}\right)-\left(\frac{1-3\mu}{2\mu}\right)+\frac{e^2}{3(1-\mu)}\left(\frac{1-3\mu}{2\mu}\right)\Big]\sigma_1+\Big[-\frac{19}{24}+\frac{17\mu}{4}+\frac{239\varepsilon_1}{24}-\frac{869\mu\varepsilon_1}{24}-\frac{5\mu\varepsilon_2}{6}+\frac{13e^2}{3}+\frac{79e^2}{24(1-\mu)} \\
& +\frac{2}{3}\left(\frac{1-2\mu}{2\mu}\right)-\frac{e^2}{2}\left(\frac{1-2\mu}{2\mu}\right)-\frac{10\varepsilon_1}{3}\left(\frac{1-2\mu}{2\mu}\right)-\frac{\mu}{3}\left(\frac{1-2\mu}{2\mu}\right)+5\mu\varepsilon_1\left(\frac{1-2\mu}{2\mu}\right)\Big]\sigma_2+\Big[\frac{287}{48}+\frac{47\mu}{12}-\frac{2695\varepsilon_1}{48}-\frac{35\mu\varepsilon_1}{12} \\
& -\frac{50\mu\varepsilon_2}{3}-\frac{2411e^2}{96}+\frac{17e^2}{6(1-\mu)}-\frac{2}{3(1-\mu)}+\frac{\mu}{(1-\mu)}+\frac{10\varepsilon_1}{3(1-\mu)}+\frac{5\mu^2}{3(1-\mu)}-\frac{5e^2}{4}\left(\frac{\mu}{(1-\mu)}\right)-\frac{35\varepsilon_1}{3}\left(\frac{\mu}{(1-\mu)}\right)+ \\
& \frac{20\mu\varepsilon_2}{3(1-\mu)}\Big]\sigma_1'+\Big[-\frac{307}{48}-\frac{25\mu}{12}+\frac{251\varepsilon_1}{12}-\frac{85\mu\varepsilon_1}{24}-\frac{85\mu\varepsilon_2}{24}+\frac{979e^2}{96}+\frac{2}{3(1-\mu)}-\frac{7\mu}{3(1-\mu)}-\frac{13e^2}{12(1-\mu)}+\frac{\mu e^2}{4(1-\mu)} \\
& -\frac{10\varepsilon_1}{3(1-\mu)}+\frac{95\mu\varepsilon_1}{6(1-\mu)}+\frac{\mu^2}{(1-\mu)}+\frac{35\mu\varepsilon_2}{6(1-\mu)}\Big]\sigma_2'+\Big[\frac{115}{12}-\frac{1051\varepsilon_1}{12}+\frac{11\mu}{6}-\frac{40\mu\varepsilon_2}{3}-\frac{569e^2}{24}+\frac{43e^2}{6(1-\mu)}- \\
& \frac{8}{3(1-\mu)}+\frac{14}{3}\left(\frac{\mu}{1-\mu}\right)+\frac{20}{3}\left(\frac{\varepsilon_1}{1-\mu}\right)-\frac{5}{3}\left(\frac{\mu\varepsilon_1}{1-\mu}\right)-\frac{5\mu^2}{3(1-\mu)}+\frac{13\mu\varepsilon_1}{6}-\frac{5\mu e^2}{4(1-\mu)}+\frac{20\mu\varepsilon_2}{3(1-\mu)}\Big]A_4\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\Omega_{yy}^{0} = {} & 3-12\varepsilon_1+12\mu\varepsilon_1-12\mu\varepsilon_2-5e^2+28e^2\varepsilon_1-\frac{2e^2}{1-\mu}+\frac{10e^2\varepsilon_1}{1-\mu}+\left[-\frac{65}{2}+18\mu+309\varepsilon_1-309\mu\varepsilon_1+69\mu\varepsilon_2+\right.\\
& \left.4\left(\frac{1-3\mu}{2\mu}\right)-20\varepsilon_1\left(\frac{1-3\mu}{2\mu}\right)+20\mu\varepsilon_1\left(\frac{1-3\mu}{2\mu}\right)+\frac{1263e^2}{4}-3e^2\left(\frac{1-3\mu}{\mu}\right)+\frac{51e^2}{1-\mu}\right]\sigma_1+\left[\frac{67}{2}-336\varepsilon_1-23\mu+\frac{987\mu\varepsilon_1}{2}-\right.\\
& \left.4\left(\frac{1-2\mu}{2\mu}\right)+20\varepsilon_1\left(\frac{1-2\mu}{2\mu}\right)+2(1-2\mu)-10\varepsilon_1(1-2\mu)-\frac{181e^2}{2}+6e^2\left(\frac{1-2\mu}{2\mu}\right)-\frac{63e^2}{1-\mu}-51\mu\varepsilon_2+20\mu\varepsilon_2\left(\frac{1-2\mu}{2}\right)\right]\sigma_2+\\
& \left[-\frac{23}{2}+69\varepsilon_1+3\mu-69\mu\varepsilon_1+4\left(\frac{1}{1-\mu}\right)-10\mu\left(\frac{1}{1-\mu}\right)-20\varepsilon_1\left(\frac{1}{1-\mu}\right)-\frac{30\mu\varepsilon_1}{1-\mu}+\frac{45e^2}{2}+\frac{127\mu\varepsilon_2}{2}-\frac{20\mu\varepsilon_2}{1-\mu}\right]\sigma_1^{'}+\\
& \left[\frac{13}{2}-\frac{71\varepsilon_1}{2}+\frac{71\mu\varepsilon_1}{2}+\frac{2}{1-\mu}-\frac{6\mu}{1-\mu}+\frac{10\varepsilon_1}{1-\mu}-\frac{20\mu\varepsilon_1}{1-\mu}-\frac{6e^2}{1-\mu}-\frac{33e^2}{4}-31\mu\varepsilon_2-\frac{10\mu\varepsilon_2}{1-\mu}\right]\sigma_2^{'}+\left[-47+415\varepsilon_1+\frac{63\mu}{2}-\right.\\
& \left.\frac{725\mu\varepsilon_1}{2}-\frac{8}{1-\mu}+\frac{10\mu}{1-\mu}+\frac{40\varepsilon_1}{1-\mu}-\frac{90\mu\varepsilon_1}{1-\mu}+\frac{42e^2}{1-\mu}+\frac{357e^2}{4}+\frac{5\mu e^2}{1-\mu}+20\mu\varepsilon_2\left(\frac{2}{1-\mu}\right)+40\mu\varepsilon_2\right]A_4
\end{aligned}
$$

Appendix-II

$$
\begin{aligned}
Q = {} & \frac{1}{(1-e^2)^{\frac{1}{2}}}\left\{\frac{35}{8}-\frac{87\varepsilon_1}{8}+\frac{87\mu\varepsilon_1}{8}-\frac{53e^2}{16}-\frac{3\mu e^2}{2}-\frac{11e^2}{4(1-\mu)}+\frac{55e^2\varepsilon_1}{4(1-\mu)}+\frac{421e^2\varepsilon_1}{16}-\frac{87\mu\varepsilon_2}{8}+\frac{3\mu e^2}{4(1-\mu)}+\right.\\
& \left[\frac{-1873}{64}+\frac{293\mu}{16}+\frac{4875\varepsilon_1}{16}-\frac{4455\mu\varepsilon_1}{16}+\frac{795\mu\varepsilon_2}{16}+\frac{9885e^2}{32}+\frac{471e^2}{8(1-\mu)}+\frac{5}{4}\left(\frac{1-3\mu}{\mu}\right)-\frac{25}{4}\left(\frac{1-3\mu}{\mu}\right)\varepsilon_1+\right.\\
& \left.\frac{3}{4}(1-3\mu)-\frac{15}{4}(1-3\mu)\varepsilon_1+10\mu\varepsilon_1\left(\frac{1-3\mu}{2\mu}\right)\right]\sigma_1+\left[\frac{4351}{64}-\frac{353\mu}{16}-\frac{4899\varepsilon_1}{16}+\frac{33171\mu\varepsilon_1}{64}-\right.\\
& \frac{579\mu\varepsilon_2}{16}-\frac{1289e^2}{16}+\frac{15}{4}(1-2\mu)\varepsilon_2-\frac{2013e^2}{32(1-\mu)}-\frac{5}{4}\left(\frac{1-2\mu}{\mu}\right)+\frac{1}{2}(1-2\mu)-\frac{25}{4}(1-2\mu)\varepsilon_1+\\
& \left.\frac{55}{4}\left(\frac{1-2\mu}{\mu}\right)\varepsilon_1\right]\sigma_2+\left[\frac{-13}{4}-\frac{219\mu}{64}-\frac{885\mu\varepsilon_1}{16}+\frac{117\varepsilon_1}{2}-\frac{14737\mu\varepsilon_2}{64}-\frac{105\varepsilon_2}{4}+\frac{129e^2}{32}+\frac{12e^2}{(1-\mu)}\right.\\
& \left.+\frac{5}{2(1-\mu)}-\frac{13}{4}\left(\frac{\mu}{1-\mu}\right)-\frac{25}{2}\left(\frac{\varepsilon_1}{1-\mu}\right)+\frac{243\mu\varepsilon_1}{8(1-\mu)}-\frac{15\mu^2}{2(1-\mu)}+3\mu\left(\frac{1-3\mu}{4}\right)-\frac{55\mu\varepsilon_2}{2(1-\mu)}\right]\sigma_1^{'}+\left[\frac{1}{2}+\frac{275\mu}{64}-\frac{109\varepsilon_1}{4}+\right.\\
& \frac{145\mu\varepsilon_1}{4}+\frac{17249\mu\varepsilon_2}{64}+\frac{105\varepsilon_2}{4}+\frac{309e^2}{8}-\frac{135e^2}{8(1-\mu)}+\frac{11}{4(1-\mu)}-\frac{39}{4}\left(\frac{\mu}{1-\mu}\right)+\frac{25}{4}\left(\frac{\varepsilon_1}{1-\mu}\right)-\frac{5\mu\varepsilon_1}{(1-\mu)}+\frac{9\mu^2}{2(1-\mu)}-\\
& \left.\frac{25\mu\varepsilon_2}{4(1-\mu)}\right]\sigma_2^{'}+\left[\frac{-167}{4}+\frac{5575\varepsilon_1}{16}+27\mu-\frac{105\varepsilon_2}{8}+\frac{1225\mu\varepsilon_2}{16}+\frac{4287e^2}{64}+\frac{237e^2}{4(1-\mu)}-\frac{11}{(1-\mu)}+\frac{79}{4}\left(\frac{\mu}{1-\mu}\right)+\right.\\
& \left.\left.55\left(\frac{\varepsilon_1}{1-\mu}\right)-\frac{495\mu\varepsilon_1}{4(1-\mu)}-\frac{15\mu^2}{2(1-\mu)}-355\mu\varepsilon_1+\frac{35\mu\varepsilon_2}{(1-\mu)}\right]A_4-4(1-e^2)^{\frac{1}{2}}\right\}
\end{aligned}
$$

$$
\begin{aligned}
R = {} & \frac{7}{8}+\frac{11\varepsilon_1}{8}-\frac{51\mu\varepsilon_1}{8}-\frac{29\mu\varepsilon_2}{8}+\frac{35e^2}{8}-\frac{1115e^2\varepsilon_1}{48}-\frac{7e^2}{3(1-\mu)}+\frac{87e^2\varepsilon_1}{4(1-\mu)}-\frac{37\mu e^2}{6}+\frac{19\mu e^2}{12(1-\mu)}+\\
& \mu-\mu^2+5\mu^2\varepsilon_1+5\mu^2\varepsilon_2+\left[-\frac{3065}{192}+\frac{241\mu}{48}+\frac{2351\varepsilon_1}{48}+\frac{1205\mu\varepsilon_1}{48}-\frac{5327\mu\varepsilon_2}{48}+\frac{25}{24}\left(\frac{1-3\mu}{2\mu}\right)+\right.\\
& \left.\frac{29\varepsilon_1}{6}\left(\frac{1-3\mu}{2\mu}\right)+\frac{15\mu\varepsilon_1}{2}\left(\frac{1-3\mu}{2\mu}\right)+\frac{2291e^2}{64}+\frac{18901e^2}{288(1-\mu)}+\frac{9(1-3\mu)}{4}-\frac{81\varepsilon_1(1-3\mu)}{4}+\frac{7\mu}{3}\left(\frac{1-3\mu}{2\mu}\right)\right]\sigma_1\\
& +\left[\frac{3131}{192}-\frac{1181\varepsilon_1}{48}-\frac{559\mu}{48}+\frac{53335\mu\varepsilon_1}{192}+\frac{7}{3}\left(\frac{1-2\mu}{2\mu}\right)+\frac{85\varepsilon_1}{6}\left(\frac{1-2\mu}{2\mu}\right)-\frac{15(1-2\mu)}{4}+\right.\\
& \left.\frac{111\varepsilon_1(1-2\mu)}{4}+\frac{7525e^2}{192}-\frac{3985e^2}{72(1-\mu)}+\frac{2687\mu\varepsilon_2}{48}+\frac{45\varepsilon_2(1-2\mu)}{4}+\frac{5\mu}{3}\left(\frac{1-2\mu}{2\mu}\right)\right]\sigma_2\\
& +\left[\frac{347}{24}-\frac{1517\varepsilon_1}{48}+\frac{5027\mu}{192}+\frac{703\mu\varepsilon_1}{8}-\frac{7}{3(1-\mu)}-\frac{401\mu}{6(1-\mu)}-\frac{157\varepsilon_1}{6(1-\mu)}-\frac{129\mu\varepsilon_1}{4(1-\mu)}-\frac{5051e^2}{96}\right.\\
& \left.-\frac{3018704\mu\varepsilon_2}{3072}-\frac{85\mu\varepsilon_2}{6(1-\mu)}+\frac{3421e^2}{72(1-\mu)}-\frac{315\varepsilon_2}{4}+\frac{9\mu(1-3\mu)}{4}+\frac{70\mu\varepsilon_1}{1-\mu}\right]\sigma_1'\\
& +\left[-\frac{55}{6}+\frac{1817\varepsilon_1}{48}-\frac{387\mu\varepsilon_1}{8}+\frac{7}{3(1-\mu)}-\frac{59\mu}{6(1-\mu)}-\frac{91\varepsilon_1}{12(1-\mu)}+\right.\\
& \left.\frac{499\mu\varepsilon_1}{12(1-\mu)}-\frac{2561e^2}{72(1-\mu)}+\frac{13087e^2}{96}+\frac{3086608\mu\varepsilon_2}{3072}+\frac{3\mu\varepsilon_2}{4(1-\mu)}+\frac{419\mu}{192}+\frac{315\varepsilon_2}{4}-\frac{35\mu\varepsilon_2}{6(1-\mu)}\right]\sigma_2'+\\
& \left[-\frac{275}{24}-\frac{391\varepsilon_1}{16}+\frac{751\mu}{48}-\frac{2981\mu\varepsilon_1}{24}-\frac{28}{3(1-\mu)}-\frac{94\mu}{(1-\mu)}+\frac{67\varepsilon_1}{(1-\mu)}-\frac{1349\mu\varepsilon_1}{12(1-\mu)}+\right.\\
& \left.\frac{1601e^2}{16(1-\mu)}-\frac{4305e^2}{64}+\frac{61\mu\varepsilon_2}{3(1-\mu)}-\frac{1237\mu\varepsilon_2}{48}-\frac{315\varepsilon_2}{8}\right]A_4
\end{aligned}
$$