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On Classifying the Skull Dimensions of the Wolf by the Discriminant Function

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Abstract— This study examines the discriminant function analysis on the skull dimensions of samples of wolf skulls from northwestern Canada in four regions which include Rocky mountain males and Rocky mountain females as well as Arctic males and Arctic females. The variables that were measured in millimeters for each skull of a wolf are Y_1: palatal length, Y_2: postpalatal length, Y_3: Zygomatic width, Y_4: palatal width outside the first upper molar, Y_5: palatal width inside the second upper premolars, Y_6: width between the postglenoid foramina, Y_7: interorbital width, Y_8: least width of the braincase and Y_9: crown length of the first upper molar. We produced the discriminant function equations for the four regions and stated the rules for classifying a certain variable that depicts a skull into one of the four regions considered in the study, that is, Rocky mountain males, Rocky mountain females, Arctic males and Arctic females. In this article, we employed the classification rules to classify each of the N = 25 statement vectors such that the classification and discrimination procedure asserted that 92.0% of the original grouped cases were correctly classified and 88.0% of the cross-validated grouped cases were correctly classified. The analyses in this article were analyzed and executed with the Statistical Package for Social Sciences *(SPSS)* software version 8.0

Keywords— discriminant function, classification, wolf, region, classification rules, tolerance level

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1. Introduction

In the practical sciences there exist a category of multivariate problem that often happens in which an observation needs to be allotted in a certain optimum manner to any of several entities. For instance, in psychology of education, an applicant for admittance to a university must be allocated to specific sorts of the kind "admit," "admit provisionally," or 'not yet admitted" on the basis of a vector of quiz marks, rankings and grades. In financial organizations, a financial expert may desire to categorize loan contenders as low, moderate or high mortgage risks, predicated on the premise of some basics of specific accounting reports. In plant science taxonomy, a botanist may desire to categorize a different specimen as any of some familiar sorts of a flower. In all of these instances, the verdict maker may desire to categorize from simple functions of the statement vector, relative to complex areas in the advanced dimensional plane of the vector.

In this article, we shall consider classification rules based on an index called the linear discriminant function. "Reference [1]" defined classification as a multivariate technique for assigning new entities into groups when we identify the distribution of every population from which the new entities emerged. The characteristics of the population from which the new entities emerged must be assessed from history as it is the practice in multivariate statistical methods.

"Reference [2]" asserts that discriminant analysis is a scientific method employed usually to differentiate between sets of populations Ω_i and to govern the means to assign new entities into various groups. The technique is used in circumstances where the different groups are identified as a priori. The purpose of the scientific technique is to classify an entity, or numerous entities, into the various identified groups. For example, in credit counting, a financial institution is aware from previous knowledge that there exist upright customers (who pay back their loan devoid of every glitches) and unscrupulous customers (who exhibited problems in refunding their credit). However, once a new client requests for a credit, the financial institution would have to agree on granting the loan or not to grant the mortgage. Previous proceedings of the financial institution offers two sets of data: multivariate data y_i on the types of customers such as level of education, marital status, age, the sum of the loan, salary, etc. A new observation y is the new client with similar variables. The rule of discrimination must classify the borrower into any of the groups such that the discriminant analysis would assess the peril of a potential "unscrupulous decision".

This study employed secondary data on the skull dimensions of a wolf and the data reflected the sexual and environmental differences in the skull measurement of the wolf *Canis lupus*. The novel study embodies samples of wolf skulls from northwestern Canada in four provinces, but for uncomplicatedness in the workout, we only provide the data for the lesser samples from the Rocky Mountain and the Arctic Archipelago. The variables below were measured in millimeters for each skull of a wolf.

Y_1 = palatal length

Y_2 = postpalatal length

Y_3 = Zygomatic width

Y_4 = palatal width outside the first upper molars

Y_5 = palatal width inside the second upper molars

Y_6 = width between the postglenoid foramina

Y_7 = interorbital width

Y_8 = least width of the braincase

Y_9 = crown length of the first upper molar length

The study is also centered on the following objectives (i) To state the rules for classifying a certain skull into one of the four groups: Rocky Mountain males, Rocky Mountain females, and Arctic Archipelago males as well as Arctic Archipelago females (ii) To employ the rules in objective (i) above to classify each of the N = 25 statement vectors into a clutch population and to produce a matrix of quantities or numbers of accurate and inaccurate classifications (iii) To compare the linear discriminant coefficients of male and female Rocky Mountain wolves, male and female Arctic Archipelago wolves, and equally, Rocky Mountain and Arctic wolves of each gender by assuming a common variance matrix estimated by **V**.

The rest of this study is structured as follows. In Section II, we present the theoretical literature review and empirical framework associated with the study. In section III, we present the data and the methodology used in the study which deals with the discriminant function in classifying observations emerging from skull dimensions of a wolf given in millimeters. In Section IV, we dealt with the analysis of the data and discussion of results. We present the conclusion and direction for upcoming studies in discrimination and classification as techniques in multivariate statistical methods in section V.

2. Literature Review

A. Theoretical Literature

Canis lupus, the wolf is the leading affiliate of the dog (Canidae) kinfolk and it is one of the glowing investigated predators on earth. *Canis lupus* exists in parks that plays a role of kinfolk units for its members. The wolf is characterized by a pair of stuck adults who participate profoundly in the upbringing of their descendants and the offspring incline to stay with their paternities for several years before separating [3].

Archaeologically, stretching through almost totally of North America and Eurasia, the *canis lupus* spread was circumpolar beyond 15^0-20^0 N latitude. Wolves are the only animals that have linger next to humans' vis-à-vis their geographical assortment. For this reason, *canis lupus* has been exposed and is still expose to an extensive variation in of landscapes and weathers in its evolutionary antiquity, beginning with the arctic tundra as well as boreal taiga to temperate coniferous woodlands as well as prairie savanna grassland. These are extracts from [4].

Canis lupus, the wolf exhibits multifaceted group behaviors thereby establishing supremacy orders. These dominant hierarchies are at times one for both sex, voluptuous dimorphism as well as dissection of reproductive and hunting toil has been detected continually among wolves. These supremacy hierarchies exhibited by wolves operates alongside other complex behaviors characterized by ferocious territoriality, intra precise rivalry and fierceness. They are environmental elements dispersed over massive ranges, focusing in the killing of huge prey. These are extracts from [5].

"Reference [6]"viewed multivariate technique as a key area in statistical methods and has elucidated difficulties in classifications of multivariate data. Investigations involving discrimination analysis as well as logistic regression have served appropriately as implements that are employed for classification and forecast. "Reference [6]" further stated that classification into one of numerous observations is discriminant analysis, or classification. According to [6] Fisher's methodology to discriminant analysis is parametric and depend on a multivariate normality assumptions for optimality. Therefore, Fisher's methodology to discriminant analysis may not be as much efficient on more genuine classes of difficulties. Numerous approaches for discriminant analysis have been suggested. Variations amid methods ascend due to variability in continuous distributions as well as discrete distributions assumptions made on the variables of interest that describes every object to be classified. Discriminant analysis methods established on the assumption of normality are the utmost commonly used in multivariate statistical methods in practice.

In constructing a procedure of classification, it is desired to minimize the probability of misclassification, or, more specifically, it is desired to minimize on the average the bad effects of misclassification [7].

B. Theoretical Literature

A study by [8] to provide a more realistic and reliable way of placing Nigerian students seeking admission into Nigerian University system, using discriminant function analysis and to provide a quantitative analysis of a discriminant function analysis approach to predict student's admission scores into a university system. The study employed university mandatory examination (UME) and aptitude test score of students in various faculties. The study showed that the linear function established a hit ratio of 83% which successfully predicted the student admission scores. The study also revealed an apparent error rate of 17% which explains the probabilities of misclassification.

A discriminant function analysis was executed on the classification of students predicated on the premise of academic performance and the discrimination rule was attained. The investigation showed that the linear discriminant function re-classified some students on the basis of their academic performances. The discrimination rule discriminated four students from Statistics department to Computer Science department, and five students from Computer Science department to Statistics department established on the grades they obtained in four courses the students offered together [9].

"Reference [10] addresses the problem of statelessness by implementing a nonparametric kernel discriminant function to classify the stateless populations in Kenya and compare the performance of the nonparametric kernel discriminant method with existing methods by their respective classification rates. Nonparametric discriminant functions have demonstrated to be more robust and suitable, particularly once there is an auxiliary information that would be used to intensify accuracy. The findings from this study revealed that nonparametric discriminant classifiers offer a good classification method for classifying the stateless communities in Kenya. This is because they exhibit lower classification rates compared to the parametric techniques such as the quadratic discriminant function and the linear discriminant function. Moreover, the findings revealed that specific relationships in characteristics that occur in these populations that are around the stateless community of Pemba, such that, the community of Pemba was classified as Giriama or Rabaiin communities which they appear to possessed a sturdy connection.

A study on managing staff appraisal used multivariate discriminant analysis and the investigation revealed that the discriminant function has suitably evaluated and classifies about 67% of the entities that are involved in the analysis.

The study fashioned out two discriminant functions, as the dependent variable has three classes. The numerical results showed that the first discriminant function is more critical than the second discriminant function, since 77% of the variance amid the groups is explained by the first discriminant function, while 23% of this variance is explained by the second discriminant function [11].

3. Data and Methodology

A. Data

In this study, we obtained the data from [12]. The data is hinged on geographical and sexual differences in the skull dimensions of the wolf (*Canis lupus)* which we described already in section I of this article. We are greatly indebted to Dr. Pierre Jolicoeur for permitting “Ref. [12]” to reproduce the data in his text. We also present the data in this article in Table 1. The variables $Y_{1,...,}$ Y_2 are as defined in section I of the article. We shall use SPSS discriminant function analysis software to analyze the data. The analysis is executed in section IV of this article.

B. Methodology

In this study, we employed discriminant function analysis in the classification of the skull dimensions of a wolf from two different regions in Canada as shown in the data of Table 1. The regions considered in this study consists of male and female wolves respectively. From Table 1, one could observe that the data is divided into four groups. This means that the classification is the kind that involves several groups. Therefore, we would provide the rules for classifying a specified skull keen on any of the four clusters (Rocky mountain males, Rocky mountain females, Arctic males and Arctic females).

In this study, the mean vector and covariance matrix are swapped by their standard estimators in equations (1) and (2) respectively.

$$\hat{\gamma}_j = \bar{y}_j \tag{1}$$

$$V = \frac{1}{W-l}\sum_{j=1}^{l} D_j \tag{2}$$

The estimators of equations (1) and (2) are defined in terms of the sample mean vectors $\bar{y}_j$ and the sums of squares and products matrix D_j for the j^{th} cluster or group. We calculate the linear discriminant scores when y is the new element of a new derivation by equation (3).

$$Z_{ij} = y^{/}V^{-1}\left(\bar{y}_i - \bar{y}_j\right) - \frac{1}{2}\left(\bar{y}_i + \bar{y}_j\right)^{/} V^{-1}\left(\bar{y}_i - \bar{y}_j\right) \tag{3}$$

The classification rule follows as in the statement below.

Assign y to population I if $Z_{ij} > 0$ for all $j \neq i$

From the foregoing classification rule above, one would quickly notice that $Z_{ij} = -Z_{ji}$ and that any l-1linearly independent Z_{ij} form the basis of the complete set of the statistics if $l - 1 < q$. If $q < l$-1the space of the Z_{ij} will have rank q, and the classification rule can be derived in terms of q scores. In this study, we have l = 4 and we let q be two or more. The distinct discriminant statistics are

$$Z_{12} = y^{/}V^{-1}\left(\bar{y}_1 - \bar{y}_2\right) - \frac{1}{2}\left(\bar{y}_1 + \bar{y}_2\right)^{/} V^{-1}\left(\bar{y}_1 - \bar{y}_2\right)$$

$$Z_{13} = y^{/}V^{-1}\left(\bar{y}_1 - \bar{y}_3\right) - \frac{1}{2}\left(\bar{y}_1 + \bar{y}_3\right)^{/} V^{-1}\left(\bar{y}_1 - \bar{y}_3\right)$$

Table 1. Skull dimensions of the wolf in millimetres

Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8	Y_9
			Rocky	**Mt.**	**Males**			
126	104	141	81.0	31.8	65.7	50.9	44.0	18.2
128	111	151	80.4	33.8	69.8	52.7	43.2	18.5
126	108	152	85.7	34.7	69.1	49.3	45.6	17.9
125	109	141	83.1	34.0	68.0	48.2	43.8	18.4
126	107	143	81.9	34.0	66.1	49.0	42.4	17.9
128	110	143	80.6	33.0	65.0	46.4	40.2	18.2
			Rocky	**Mt.**	**Females**			
116	102	131	76.7	31.5	65.0	45.4	39.0	16.8
120	103	130	75.1	30.2	63.8	44.4	41.1	16.9
116	103	125	74.7	31.6	62.4	41.3	44.2	17.0
			Arctic		**Males**			
117	99	134	83.4	34.8	68.0	40.7	37.1	17.2
115	100	149	81.0	33.1	66.7	47.2	40.5	17.7
117	106	142	82.0	32.6	66.8	44.9	38.2	18.2
117	101	144	82.4	32.8	67.5	45.3	41.5	19.0
117	103	149	82.8	35.1	70.3	48.3	43.7	17.8
119	101	143	81.5	34.1	69.1	50.1	41.1	18.7
115	102	146	81.4	33.7	66.4	47.7	42.0	18.2
117	100	144	81.3	37.2	66.8	41.4	37.6	17.7
114	102	141	84.1	31.8	67.8	47.8	37.8	17.2
110	94	132	76.9	30.1	62.1	42.0	40.4	18.1
			Arctic		**Females**			
112	94	134	79.5	32.1	63.3	44.9	42.7	17.7
109	91	133	77.9	30.6	61.9	45.2	41.2	17.1
112	99	139	77.2	32.7	67.4	46.9	40.9	19.3
112	99	133	78.5	32.5	65.5	44.2	34.1	17.5
113	97	146	84.2	35.4	68.7	51.0	43.6	17.2
107	97	137	78.1	30.7	61.6	44.9	37.3	16.5

Table 2. Group statistics for skull dimensions of the wolf

Region	Variables	Mean	Std dev
Rocky Mountain males	Y_1	126.5	1.2
	Y_2	108.2	2.5
	Y_3	145.2	4.9
	Y_4	82.1	2.0
	Y_5	33.6	1.0
	Y_6	67.3	1.9
	Y_7	49.4	2.2
	Y_8	43.2	1.8
	Y_9	18.2	0.2
Rocky Mountain females	Y_1	117.3	2.3
	Y_2	102.7	0.58
	Y_3	128.7	3.2
	Y_4	75.5	1.1
	Y_5	31.1	0.8
	Y_6	63.7	1.3
	Y_7	43.7	2.1
	Y_8	41.4	2.6

	Y_9	16.9	0.1
Arctic males	Y_1	115.8	2.5
	Y_2	100.8	3.1
	Y_3	142.4	5.6
	Y_4	81.7	1.9
	Y_5	33.5	1.9
	Y_6	67.1	2.1
	Y_7	45.5	3.2
	Y_8	39.9	2.2
	Y_9	17.9	0.6
Arctic females	Y_1	110.8	2.3
	Y_2	96.2	3.12
	Y_3	137.0	5.0
	Y_4	79.2	2.5
	Y_5	32.3	1.7
	Y_6	64.7	2.9
	Y_7	46.2	2.5
	Y_8	39.9	3.6
	Y_9	17.5	0.9
Totals	Y_1	117.4	6.1
	Y_2	101.7	5.0
	Y_3	140.1	7.1
	Y_4	80.6	2.9
	Y_5	32.9	1.8
	Y_6	66.2	2.5
	Y_7	46.4	3.2
	Y_8	40.9	2.8
	Y_9	17.8	0.7

$$Z_{23} = y^{/}V^{-1}\left(\bar{y}_2 - \bar{y}_3\right) - \frac{1}{2}\left(\bar{y}_2 + \bar{y}_3\right)^{/} V^{-1}\left(\bar{y}_2 - \bar{y}_3\right) \tag{4}$$

$$Z_{24} = y^{/}V^{-1}\left(\bar{y}_2 - \bar{y}_4\right) - \frac{1}{2}\left(\bar{y}_2 + \bar{y}_4\right)^{/} V^{-1}\left(\bar{y}_2 - \bar{y}_4\right)$$

The classification rule is defined as follows.

Classify y as from

Population 1 if $Z_{12} > 0$ and $Z_{13} > 0$

Population 2 if $Z_{12} < 0$ and $Z_{13} > Z_{12}$

Population 3 if $Z_{13} < 0$ and $Z_{12} > Z_{13}$

Population 4 if $Z_{13} < 0$ and $Z_{12} > Z_{14}$

These are extracts from “Ref. [12]”.

“Reference [13]” in a related development have expressed the multi-cluster classification stated above in terms of the minimum Mahalanobis squared distance computed from uncorrelated linear compounds of the new variates. This is given by equation (5) below.

$$M_i^2 = \left(y - \bar{y}_i\right)' V^{-1} \left(y - \bar{y}_i\right) \quad (5)$$

of the new observation from the mean of the i^{th} sample given in (6) below.

Assign y to population i if $M_i^2 = \min\left\{M_1^2, \ldots, M_l^2\right\}$ (6)

The equivalence of the rules follows from the relation in equation (7)

$$Z_{hj} = -\frac{1}{2} M_h^2 + \frac{1}{2} M_j^2 \quad (7)$$

so that $Z_{ij} > 0$ for all $j \neq i$ is satisfied if $M_i^2 < M_j^2$ for all $j \neq i$.

Table 3. Tests of equality of group means

Variables	Wilk's Lambda	F	df1	df2	sig.
Palatal length (Y_1)	0.114	54.355	3	21	.000
Postpalatal length (Y_2)	0.271	18.794	3	21	.000
Zygomatic width (Y_3)	0.459	8.240	3	21	.001
Palatal width outside the 1st upper molars(Y_4)	0.439	8.949	3	21	.001
Palatal width inside the 2nd upper premolars (Y_5)	0.756	2.263	3	21	.111
Width between the postglenoid foramina (Y_6)	0.695	3.075	3	21	.050
Interorbital width (Y_7)	0.654	3.698	3	21	.028
Least width of the braincase (Y_8)	0.750	2.329	3	21	.104
Crown length of the 1st upper molar (Y_9)	0.664	3.545	3	21	.032

The discriminant function analysis is employed in this study for the fact that it builds a predictive model for group membership such that the model involves a discriminant function founded on a linear combination of criterion variables and these criterion variables gives the finest discrimination amongst clusters or groups.

In this article, the purpose of employing discriminant function analysis is to extremely isolate the groups in order to define the maximum parsimonious fashion to separating groups as well as dispose of variables which are diminutively interconnected to group differences. We are also concerned in this study with the connection between a cluster of independent variables and a categorical variable. We are also concerned in determining how many measurements we would require to necessitate this connection. By means of this relationship, we can envisage a classification built on the independent variables or evaluate how glowing the independent variables isolate the categories in the classification process. It is pertinent to note that discriminant function analysis is similar to regression analysis. A discriminant score is computed on the weighted combination of the independent variables. To illustrate the foregoing assertion, we give a multiple linear regression model in the equation (8) below.

$$g_i = a_0 + a_1 y_1 + a_2 y_2 +, \ldots, + a_n y_n \quad (8)$$

where i = 1,…,n. This follows from equation (8) that the term g_i is the predicted score of the regression relation and it is equivalent to the discriminant score in discriminant function analysis, while y is the predictor in the regression relation and a is the discriminant coefficient.

4. Results and Discussions

In this section, we present the analysis of the data in Table 1 using the discriminant function analysis presented in section III of this article. We employed SPSS software on discriminant function analysis to analyze the data. Table 2 presents statistics for the data where the means and standard deviation for the data is demonstrated for the regions presented in Table 1. The Y variables in all tables of this article are as defined in section I. The information in Table 2 is employed to determine the information in Table 3. The information presented in Table 3 is with regards to the test of equality of group means using the Wilk's lambda statistics and the F test statistics. From Table 3, one would observe that for all values of the Wilk's lambda statistics given in column two are all significant for all the variable on skull dimensions of the wolf presented in Table 1 for all the four regions. Also, the F test statistics is given in column three of Table 3. From the results of the F test statistics one would observe that all values of the F test statistics given in column three are all significant for all the variables on skull dimensions of the wolf presented in Table 1 for all the four regions. These variables are $Y_{1:}$ palatal length, $Y_{2:}$ postpalatal length, $Y_{3:}$ Zygomatic width, $Y_{4:}$ palatal width outside the first upper molar, $Y_{5:}$ palatal width inside the second upper premolars, $Y_{6:}$ width between the postglenoid foramina, $Y_{7:}$ interorbital width, $Y_{8:}$ least width of the brain and $Y_{9:}$ crown length of the first upper molar.

The covariance matrix for all the nine variables defining the skull dimensions of the skull of all four regions presented in Table 1, that is, Rocky mountain males, Rocky mountain females, Arctic males and arctic females was obtained using the SPSS software version 8.0. In this study, we employed only the upper triangular part of the covariance matrix to determine the classification function coefficients presented in Table 6. We stated in section III of this article that one of the purpose of discriminant function analysis to remove variables which are bantam connected to group differences. The within samples covariance matrix V is

4.8	3.1	4.6	1.6	2.0	3.2	1.8	0.3	0.5
3.1	7.9	7.1	1.7	1.6	3.5	2.2	-1.3	0.4
4.6	7.1	7.1	5.1	4.2	7.2	9.8	5.0	0.3
1.5	1.6	26.5	4.3	1.9	2.9	1.9	1.1	-0.4
2.0	1.6	4.2	1.9	2.7	2.5	0.5	0.6	0.0
3.2	3.7	7.6	2.9	2.5	5.2	3.7	1.3	0.4
1.8	2.2	9.8	1.8	0.5	3.7	7.6	3.1	0.3
0.3	-1.3	5.0	1.1	0.6	1.2	3.1	6.6	0.5
0.5	0.4	0.3	-0.4	0.0	0.4	0.3	0.5	0.4

In the analysis of the data in Table 1 by the SPSS software, six variables were removed in the analysis and they include $Y_{2:}$ postpalatal length, $Y_{3:}$ Zygomatic width, $Y_{5:}$ palatal width inside the second upper premolars, $Y_{6:}$ width between the postglenoid foramina, $Y_{7:}$ interorbital width, $Y_{8:}$ least width of the brain and Y_9. The variable that were in the analysis are three and they include $Y_{1:}$ palatal length, Y4: palatal width outside the first upper molar and $Y_{9:}$ crown length of the first upper molar. The variables passed through an iteration process by the software to remove the variables so removed and to retain the variables so retained in the analysis. The iteration process went throgh three steps and at each step, the variable that diminishes the general Wilks' lambda statistics is entered.

In the first step, the variable that minimizes the general or overall Wilk's lambda statistics is $Y_{1:}$palatal length, with a tolerace of 1.0 which is equivalent to 100% and once the tolerance level of the variable is 100% after the iteration, the variable minimizes the overall Wilk's lambda statistics and it is entered. In the second step of the iteration process, two variables were retained. These variables are $Y_{1:}$palatal length and $Y_{4:}$ palatal width outside the first upper molar with equal tolerance level of 0.876 respectively and Wilk's lambda statistics of 0.439 for the variable $Y_{1:}$palatal length and 0.114 for the variable $Y_{4:}$ palatal width outside the first upper molar.

Table 4. Pairwise groups comparisons

		Rocky	Rocky		
Step	Region	Mountain Males	Mountain Females	Arctic Males	Arctic Females
1	Rocky				
	Mt.	F:	F:35.1	F:89.6	F:153.7
	Males	Sig:	Sig:0.0	Sig:0.0	Sig:0.0
	Rocky				
	Mt.	F:35.1	F:	F:1.1	F:17.6
	Females	Sig:0.0	Sig:	Sig:0.3	Sig:0.0
	Arctic	F:89.6	F:1.1	F:	F:19.3
	Males	Sig:0.0	Sig:0.0	Sig:	Sig:0.0
	Arctic	F:153.7	F:17.6	F:19.3	F:
	Females	Sig:0.0	Sig:0.0	Sig:0.0	Sig:
2	Rocky				
	Mt.	F:	F:19.9	F:47.3	F:75.2
	Males	Sig:	Sig:0.0	Sig:0.0	Sig:0.0
	Rocky				
	Mt.	F:19.9	F:	F:13.7	F:17.3
	Females	Sig:0.0	Sig:	Sig:0.0	Sig:0.0
	Arctic	F:47.3	F:13.7	F:9.5	F:
	Males	Sig:0.0	Sig:0.3	Sig:	Sig:0.0
	Arctic	F:75.2	F:17.3	F:9.5	F:
	Females	Sig:0.0	Sig:0.0	Sig:0.001	Sig:
3	Rocky				
	Mt.	F:	F:14.6	F:37.2	F:52.9
	Males	Sig:	Sig:0.0	Sig:0.0	Sig:0.0
	Rocky				
	Mt.	F:14.6	F:	F:21.8	F:21.6
	Females	Sig:0.0	Sig:	Sig:0.0	Sig:0.0
	Arctic	F:37.2	F:21.8	F:	F:6.0
	Males	Sig:0.0	Sig:0.0	Sig:	Sig:0.05
	Arctic	F:52.9	F:21.6	F:6.0	F:
	Females	Sig:0.0	Sig:0.0	Sig:0.005	Sig:

The immediate two foregoing variables are the only two variables that diminishes the overall Wilk's lambda statistics at this second stage of the iteration, hence they are entered. At the third step of the iteration process, only three variables were retained in the analysis. These variable are $Y_{1:}$palatal length and $Y_{4:}$ palatal width outside the first upper molar and Y_9: crown length of the first upper molar. These three variables have respectively 0.651, 0.672 and 0.675 tolerance levels. It is at this third stage that the iteration process attained equilibrium, that is, a situation where the tolerance levels will no longer change even when we continue the iteration process to n >3. Tolerance is the amount of a variable's variance unaccounted for by other predictor variables in the discriminant function equation. Variables with very low tolerance levels gives very diminutive information to a model and is capable of creating computational difficulties. Essentially, tolerance is about multicollinearity. < 0.40 tolerance which is equivalent to less 40% is worthy of apprehension. Also, <0.10 tolerance which is equivalent to less than 10% is indeed challenging. The respective Wilk's lambda statistics for these three variables are respectively 0.285, 0.077 and 0.051. The tolerance values 0.651, 0.672 and 0.675 respectively for $Y_{1:}$palatal length and $Y_{4:}$ palatal width outside the first upper molar and Y_9: crown length of the first upper molar minimizes the overall Wilk's lambda statistics.

Table 5. Classification Results

		Predicted	group	membership	
		Region	RMM	RMF	Total
Original	Count	RMM	6	0	6
		RMF	0	3	3
		AM	0	0	0
		AF	0	0	0
	%	RMM	100	0	100
		RMF	0	100	100
		AM	0	0	0
		AF	0	0	0
CR	Count	RMM	6	0	6
		RMF	0	3	3
		AM	0	0	0
		AF	0	0	0
	%	RMM	100	0	100
		RMF	0	100	100
		AM	0	0	0
		AF	0	0	0
		Predicted	group	membership	
		Region	AM	AF	Total
Original	Count	RMM	0	0	0
		RMF	0	0	0
		AM	9	1	10
		AF	1	5	6
	%	RMM	0	0	0
		RMF	0	0	0
		AM	90	10	100
		AF	16.7	83.3	100
CR	Count	RMM	0	0	0
		RMF	0	0	0
		AM	9	1	10
		AF	2	4	6
	%	RMM	0	0	0
		RMF	0	0	0
		AM	90	10	100
		AF	33.3	66.7	100

Table 6 gives the discriminant function coefficients and these coefficients are employed to produce the discriminant function equations for the Rocky mountain males region of the skull dimension of the wolf, Rocky mountain females region of the skull dimension of the wolf, Arctic males region of the skull dimension of the wolf as well as Arctic females region of the skull dimension of the wolf. From Table 1, one would observe that data was presented for 6 wolves' skull dimensions in nine variable for Rocky mountain males, 3 wolves' skull dimensions in nine variable for Rocky mountain females, 10 wolves' skull dimensions in nine variable for Arctic males as well as 6 wolves' skull dimensions in nine variables for Arctic females. Hence, the discriminant function equations for the four regions are given by (9), (10), (11) and (12) respectively.

$$Z_{12} = 16.17Y_1 + 17.42Y_4 + 45.97Y_9 - 2157'.51 \quad (9)$$

$$Z_{13} = 15.09Y_1 + 15.94Y_4 + 42.42Y_9 - 1847.1 \quad (10)$$

$$Z_{23} = 12.93Y_1 + 18.98Y_4 + 51.04Y_9 - 1984 \quad (11)$$

$$Z_{24} = 12Y_1 + 18.73Y_4 + 50.79Y_9 - 1854.1 \quad (12)$$

The classification rule schedules are:

Classify a wolf with observation y as

Region 1 if $Z_{12} > 0$ and $Z_{13} > 0$

Region 2 if $Z_{12} < 0$ and $Z_{13} > Z_{12}$

Region 3 if $Z_{13} < 0$ and $Z_{12} > Z_{13}$

Region 4 if $Z_{13} < 0$ and $Z_{12} > Z_{24}$

In the classification schedules, region 1 represents Rocky mountain males, region 2 represents Rocky mountain females, region 3 represents Arctic males and, region 4 represents Arctic males.

Table 6. Classification function coefficients

Regions				
	Rocky Mountain	Rocky Mountain	Arctic	Arctic
Variables	Males	Females	Males	Females
Y_1	16.171	15.094	12.931	12.002
Y_4	17.417	15.941	18.978	18.727
Y_9	45.997	42.421	51.039	50.789
Constant	-2157.51	-1847.1	-1984	-1854.1

Table 4 presents the pairwise group comparison of the analysis. In this study, pairwise group comparison refers to a multivariate statistical technique employed in this study to evaluate the relationships among pairs of means presented in Table 2, when doing region comparison with respect to the nine variables of the skull dimensions of the wolf. This technique also confirmed the retention and removal of the variables in the analysis as explained above in this section of the article. This confirmation is made possible by the pairwise group comparisons. From Table 4, one would observe that the diagonal entries of all the four groups are empty and the upper triangular entries and the lower triangular entries for all the four regions are equivalent with respect to F test value with an assertion of significance.

Table 5 presents the classification results and these classification results were made possible by the discriminant function equations (9), (10), (11) and (12) respectively as well as the classification rule schedules that follow thereof. From Table 5 one could observe that the six wolves' skull dimensions in nine variables for Rocky mountain males remain classified in their region. In a related development, the three wolves' skull dimensions in nine variables for Rocky mountain females also remain classified in their region. In another development, nine out of ten wolves' skull dimensions in nine variables for Arctic males remain classified in their region while one only was classified under Arctic females out of the ten. For six Arctic female wolves' skull dimensions in nine variables, one was classified under the region for Arctic males while the remaining five remain classified in their region of Arctic females. Illustrating the foregoing explanation in percentages, we saw that 100% of the six wolves' skull dimensions in nine variables remain classified in their region of Rocky mountain males and 100% of the three wolves' skull dimensions in nine variables remain classified in their region of Rocky mountain females. 90% of wolves' skull dimensions in nine variable out of ten wolves' for Arctic males remain classified in their region and 10% out of ten wolves' Arctic males were classified under Arctic females. In a related analogy 83.3% wolves' skull dimensions in nine variables for Arctic females remain classified in their region, while 16.7% were classified in the region of Arctic males.

Cross validation is done only for those cases in the analysis. In cross validation, each case is classified by the functions derived from all cases other than that case. In this article, the cross validation executed by the SPSS software revealed that the six wolves' skull dimensions in nine variables for Rocky mountain males remain classified in their region. In a related development, the three wolves' skull dimensions in nine variable for Rocky mountain females also remain classified in their region. In another development, nine out of ten wolves' skull dimensions in nine variables for Arctic males remain classified in their region, while one only was classified under Arctic females out of the ten. For six Arctic female wolves' skull dimensions in nine variables, two were classified under the region for Arctic males while the remaining four remain classified in their region of Arctic females. Illustrating the foregoing explanation in percentages, we saw that 100% of the six wolves' skull dimensions in nine variables remain classified in their region of Rocky mountain males and 100% of the three wolves' skull dimensions in nine variables remain classified in their region of Rocky mountain females. 90% of wolves' skull dimensions in nine variable out of ten wolves' for Arctic males remain classified in their region and 10% out of ten wolves' of Arctic males were classified under Arctic females. In a related analogy 66.7% wolves' skull dimensions in nine variables for Arctic females remain classified in their region, while 33.3% were classified in the region of Arctic males. In the overall analyses of the discrimination and classifications analogy, 92.0% of the original grouped cases were correctly classified, while 88.0% of the cross-validated grouped cases were correctly classified.

5. Conclusion

In this article, we employed the discriminant function analysis multivariate method to analyze data on samples of wolf skulls from northwestern Canada in four provinces, but for uncomplicatedness in the workout, we only employed the data for the lesser samples from the Rocky Mountain and the Arctic Archipelago. And under the Rocky mountain region, it was subdivided into Rocky mountain males and Rocky mountain females. Also, under the Arctic Archipelago region, the region was subdivided into Arctic males and Arctic females. The following variables were measured in millimeters for each skull of a wolf.

Y_1 = palatal length

Y_2 = postpalatal length

Y_3 = Zygomatic width

Y_4 = palatal width outside the first upper molars

Y_5 = palatal width inside the second upper molars

Y_6 = width between the postglenoid foramina

Y_7 = interorbital width

Y_8 = least width of the braincase

Y_9 = crown length of the first upper molar length

We produced the discriminant function equations for the four regions and stated the rules for classifying a certain variable that depicts a skull into one of the four groups: Rocky mountain males, Rocky mountain females, Arctic males and Arctic females. In this article, we employed the classification rules to classify each of the N = 25 statement vectors in section IV of this study into a clutch population and to produce a matrix of quantities or numbers of accurate and inaccurate classifications. In this study, we also compare the linear discriminant coefficients of Rocky mountain males and females with the Arctic Archipelago males and females by assuming a common variance covariance matrix estimated by **V**. The analysis revealed the variables retained in the discriminant function analysis and the variables removed in the discriminant function analysis, where the pairwise group comparison emerging from Table 4 confirmed the reason behind the retention and rejection of the variables that depicts the skull dimensions of the wolf employed in this study. We also dealt with the discriminant function equations that was used to classify the data in Table 1 where the classification and discrimination procedure asserted that 92.0% of the original grouped cases were correctly classified and 88.0% of the cross-validated grouped cases were correctly classified. It is pertinent for multivariate statistical analyst who employs the classification and discrimination methods to their researchers to bear in their minds that every entity of a population retains certain features that are comparable to different other affiliates of the group in which its relationship is recognized. Therefore, the presentation of discriminant function analysis in this study is driven by the necessity to reclassify certain observations who by one purpose or the other fitted to the incorrect group or groups as with the regions considered in this investigation, that is, the Rocky mountain males region of the skull dimension of the wolf, the Rocky mountain females region of the skull dimension of the wolf, the Arctic males region of the skull dimension of the wolf as well as the Arctic females region of the skull dimension of the wolf as it is presented in Table 1 of this study. The necessity for discrimination function analyses rises when there exits some observations in a group whose appearances are meaningfully at variant from the overall characteristics of participants establishing the larger amount of members in a cluster. We hope that the classification and discrimination multivariate statistical method employed in this study would draw the attention of captains of industries, organizations, the agricultural sector, the education sector to adopt and deploy the multivariate technique in reclassifying certain individuals who by any aim or the other fit to an erroneous group or groups. Multivariate statistical researchers specialized in discriminant function analysis may wish to explore the application of the method invoving more than four groups and with variables $y>9$.

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